ST. BRIGID'S
COLLEGE

Mathematics Specialist Units 3 & 4
Test 3 2017

Section 1 Calculator Free

Vectors in Two & Three Dimensions and Systems of Equations

STUDENT'S NAME: Solutions

DATE Term 2 Week 1 **TIME:** 20 minutes **MARKS:** 19

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet. Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

1. **(4+2=6 marks)**

(a) Solve the following system of equations.

$$2x - y + 2z = 1$$
$$x + y - 2z = 2$$
$$x - 2y + 4z = -1$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 1 & 1 & -2 & 2 \\ 1 & -2 & 4 & -1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

✓

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 1 & 1 & -2 & 2 \\ 0 & 3 & -6 & 3 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \leftarrow R_2 - R_3 \end{array}$$ ✓

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & -3 & 6 & -3 \\ 0 & 3 & -6 & 3 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \leftarrow R_1 - 2R_2 \\ R_3 \end{array}$$

✓

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \leftarrow R_2 - R_3 \end{array}$$

$\Rightarrow$ infinitely many solutions ✓

(b) Hence explain what the equations and their solutions represent in space.

3 planes intersect in a line ✓✓

6

2. **(6 marks)**

Determine the coordinates of the point of intersection of the line

L: $x + 1 = \dfrac{y-2}{4} = z - 3$ with the plane

Π: $\tilde{r}.\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 8$

L: Let $x+1 = \dfrac{y-2}{4} = z-3 = \lambda$

$\Rightarrow$ $x = \lambda - 1$

$y = 4\lambda + 2$ ✓

$z = \lambda + 3$ ✓

intersection of L + Π $\Rightarrow$

$$\begin{pmatrix} \lambda - 1 \\ 4\lambda + 2 \\ \lambda + 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 8$$

$2(\lambda - 1) - 1(4\lambda + 2) + 3(\lambda + 3) = 8$

$2\lambda - 2 - 4\lambda - 2 + 3\lambda + 9 = 8$ ✓✓

$\lambda + 5 = 8$

$\lambda = 3$

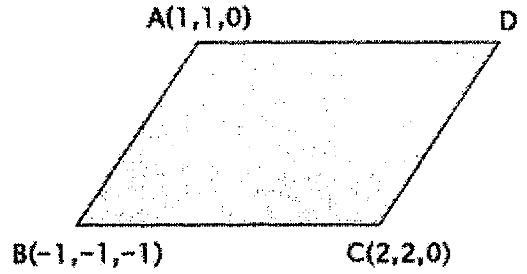
intersection point $(3-1,\ 4\times3+2,\ 3+3)$

$= (2, 14, 6)$ ✓✓

6

3. **(3+4=7 marks)**

The diagram below shows a parallelogram ABCD .



Determine

(a) the coordinates of D

Common sense approach:
$(1+3, 1+3, 0+1)$
$= (4, 4, 1)$

OR

Let $D = (x, y, z)$

$\vec{BA} = \vec{CD}$

$\vec{BO} + \vec{OA} = \vec{CO} + \vec{OD}$

$\langle 1, 1, 1\rangle + \langle 1, 1, 0\rangle = \langle -2, -2, 0\rangle + \langle x, y, z\rangle$

$\langle 2, 2, 1\rangle = \langle x-2, y-2, z\rangle$

$\Rightarrow x = 4$ ✓
$y = 4$
$z = 1$ ✓

$\Rightarrow D = (4, 4, 1)$ ✓

(b) the area of the parallelogram ABCD.

Area of parm ABCD $= |\vec{BA} \times \vec{BC}|$ ✓

$\vec{BA} = \langle 2, 2, 1\rangle$
$\vec{BC} = \vec{BO} + \vec{OC}$
$= \langle 1, 1, 1\rangle + \langle 2, 2, 0\rangle$
$= \langle 3, 3, 1\rangle$ ✓

$$\begin{matrix} i & j & k \\ 2 & 2 & 1 \\ 3 & 3 & 1 \end{matrix}$$

$\vec{BA} \times \vec{BC} = \langle 2-3, 2-3, 6-6\rangle$ ✓
$= \langle -1, -1, 0\rangle$ ✓

Area $= |\langle -1, -1, 0\rangle| = \sqrt{1+1+0} = \sqrt{2}$ sq. units

7

OR using Trig formula

End of Section 1

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Mathematics Specialist Units 3 & 4
Test 3 2017

Section 2 Calculator Assumed

Vectors in Two & Three Dimensions and Systems of Equations

STUDENT'S NAME: ______________________

DATE: Term 2 Week 1 **TIME:** 35 minutes **MARKS:** 31

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet retained from Section 1.

Special Items: Drawing instruments, templates, three calculators, notes on two sides of a single A4 page Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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4. **(4+2=6 marks)**

(a) Determine the vector equation of sphere $\prod 1$

$\prod 1$: $x^2+y^2+z^2+2x+2y+2z+2=0$

$(x^2+2x+1^2)+(y^2+2y+1^2)+(z^2+2z+1^2) = -2+1^2+1^2+1^2$

$(x+1)^2+(y+1)^2+(z+1)^2 = 1$ ✓✓

centre $(-1,-1,-1)$ radius $=1$

vector equation ; $|\underset{\sim}{r} - \langle -1,-1,-1\rangle| = 1$ ✓✓

(b) Given that the vector equation of another sphere $\prod 2$ is $\prod 2$: $\left|\tilde{r}-\left\langle -\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle\right|=1$

Calculate the distance between the centres of the two spheres $\prod 1$ and $\prod 2$

Distance between $(-1,-1,-1)$ and $(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$

$= \sqrt{(-\frac{1}{2}+1)^2+(-\frac{1}{2}+1)^2+(-\frac{1}{2}+1)^2} = \sqrt{(\frac{1}{2})^2+(\frac{1}{2})^2+(\frac{1}{2})^2}$ ✓ 6

$= \frac{\sqrt{3}}{2}$ units ✓

5. **(6+3=9 marks)**

Consider the system of equations:

$$\begin{aligned} -4x + y + 7z &= -6 \\ 2x + y - 3z &= 4 \\ x + 2y + kz &= m \end{aligned}$$

$$\left[\begin{array}{ccc|c} -4 & 1 & 7 & -6 \\ 2 & 1 & -3 & 4 \\ 1 & 2 & k & m \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} -4 & 1 & 7 & -6 \\ 2 & 1 & -3 & 4 \\ 0 & -3 & -3-2k & 4-2m \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \leftarrow R_2 - 2R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} -4 & 1 & 7 & -6 \\ 0 & 3 & 1 & 2 \\ 0 & -3 & -3-2k & 4-2m \end{array}\right] \begin{array}{l} R_1 \\ R_2 \leftarrow R_1 + 2R_2 \\ R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} -4 & 1 & 7 & -6 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -2-2k & 6-2m \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \leftarrow R_2 + R_3 \end{array}$$

(a) Determine the conditions on k and m for which

(i) the system has no solutions;

$-2-2k=0$ and $6-2m \neq 0$

$k=-1$ and $m \neq 3$ ✓✓

(ii) the system has only one solution;

$k \neq -1$ and $m \in \mathbb{R}$ ✓✓

(iii) the system has an infinite number of solutions.

$k=-1$ and $m=3$ ✓✓

6

(b) In the case where the number of solutions is infinite, determine the general solution of the system of equations in Cartesian from.

Infinitely solns $\Rightarrow$ $\left[\begin{array}{ccc|c} -4 & 1 & 7 & -6 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$-4x+y+7z=-6$ + $3y+z=2$ $\Rightarrow$ $z=2-3y$

$-4x+\left(\frac{2-z}{3}\right)+7z=-6$ $3y=2-z$ $z=\frac{3x-5}{5}$

$\times 3$ $-12x+2-z+21z=-18$ $y=\frac{2-z}{3}$ ✓

$20z=-18-2+12x$

$20z=12x-20$

$z=\frac{3x-5}{5}$

Cartesian soln

$\frac{3x-5}{5}=2-3y=z$ ✓✓

6. (1+2+2+3+2+2+2+2=16 marks)

In this question, distance is in kilometres and time is in hours.

Two small drones, Drone A and Drone B are flying at steady speeds in straight lines.
At 14:00 Drone A is at the point (1 , 1 , 6) . Its position vector $\tilde{r}_1$, measured from an origin at ground level , after t hours is given by

$$\tilde{r}_1=\langle 1,1,6\rangle+t\langle 2,-2,1\rangle$$

(a) (i) Write down the velocity vector of Drone A.

$\langle 2,-2,1\rangle$ ✓

(ii) Determine the speed of Drone A.

$=\sqrt{4+4+1} = 3$ km/h ✓✓

(iii) Assuming Drone A has been flying with this velocity since 12:00,what were Drone A's coordinates at 12:00?

$t=-2$

$\underset{\sim}{r}_1=\langle 1,1,6\rangle-2\langle 2,-2,1\rangle$ ✓✓

$=\langle -3,5,4\rangle$

8

At 14:00 Drone B is at the point (4 , 7 ,7). After four hours it is at the point (16 , -29 , 11).

(b) Show that the position of Drone B after t hours is given by

$$\tilde{r}_2 = \langle 4, 7, 7\rangle + t\langle 3, -9, 1\rangle$$

Let velocity of drone B $= \langle x, y, z\rangle$

$\underset{\sim}{r}_2 = \langle 4, 7, 7\rangle + 4\langle x, y, z\rangle = \langle 16, -29, 11\rangle$

$\langle 4+4x, 7+4y, 7+4z\rangle = \langle 16, -29, 11\rangle$ ✓✓

$x = 3, \; y = -9, \; z = 1$

$\Rightarrow$ position vector of Drone B after t hrs :

$\underset{\sim}{r}_2 = \langle 4, 7, 7\rangle + t\langle 3, -9, 1\rangle$ ✓ $\frac{3}{2}$

The paths of the two drones take them directly over the top of the same building. The top of this building is 500 metres above the ground.

(c) (i) Which drone passes over the top of the building first and what time does it do so?

(Hint : The two drones are directly on top of this building when x-coordinate and y-coordinate of Drone A and Drone B are the same at some time t_A and t_B respectively)

Let t_A + t_B be the time required for Drone A + B

i.e. $1 + 2t_A = 4 + 3t_B$ x-coordinate

$1 - 2t_A = 7 - 9t_B$ y-coordinate.

$t_A = 3.75$ hrs ✓

$t_B = 1.5$ hrs. ✓ $\frac{2}{2}$

$\Rightarrow$ Drone B passes the top first at 3.30 pm.

(ii) Determine the coordinates of the top of this 500 m building.

x-coordinate $= 4 + 3t_2 = 4 + 3 \times 1.5 = 8.5$

y-coordinate $= 7 - 9t_2 = 7 - 9 \times 1.5 = -6.5$

z-coordinate = ht of the building = 500m = 0.5 km

co ordinate of the top of the building :

$(8.5, -6.5, 0.5)$ ✓✓ $\frac{2}{2}$ 7

(iii) Determine if the paths of the two drones intersect?

intersect when

$\langle 1+2t, 1-2t, 6+t \rangle = \langle 4+3t, 7-9t, 7+t \rangle$

i.e. $1+2t = 4+3t$ $\quad$ $1-2t = 7-9t$ $\quad$ $6+t = 7+t$

$t = -3$ $\quad$ $7t = 6$ $\quad$ no solution!

$t = \frac{6}{7}$

No! paths do not intersect

Not at the same place at the same time.

(d) Calculate the angle between the paths of the two drones, giving the answer to the the nearest degrees.

$$\cos\theta = \frac{\langle 2,-2,1\rangle \cdot \langle 3,-9,1\rangle}{\sqrt{4+4+1}\ \sqrt{9+81+1}}$$

$$\theta = 29.1^\circ \simeq 29^\circ$$

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End of Section 2